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ON THE NORMS OF CIRCULANT MATRICES WITH THE GENERALIZED FIBONACCI AND LUCAS NUMBERS

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Abstract. In this paper, firstly we define $n \times n$ circulant matrices $U = Circ\ (U_0, U_1, \ldots, U_{n-1})$, $V = Circ\ (V_0, \ldots, V_{n-1})$, $T = Circ\ (T_0, \ldots, T_{n-1})$ and $S = Circ\ (S_0, \ldots, S_{n-1})$, where $\{U_n\}$ and $\{V_n\}$ are generalized Fibonacci and Lucas types second order linear recurrences, $\{T_n\}$ and $\{S_n\}$ are Tribonacci sequences with different initial conditions. After we study spectral noms of these matrices and their Hadamard and Kronecker product.

Keywords: circulant matrix, generalized Fibonacci number, generalized Lucas number, matrix norm.

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1. Introduction

For $n > 0$, the well known *Fibonacci sequence* $\{F_n\}_{n=1}^{\infty}$ is defined by

$$F_{n+1} = F_n + F_{n-1},$$

where $F_0 = 0$, $F_1 = 1$ and the *Lucas sequence* $\{L_n\}_{n=1}^{\infty}$ is defined by

$$L_{n+1} = L_n + L_{n-1},$$

where $L_0 = 2$ and $L_1 = 1$.

The Fibonacci and Lucas sequences can be generalized as follows: Let p and q be positive integer. The second order linear recurrences of the Fibonacci and Lucas types are defined by the following equations:

$$\begin{aligned} U_{n+1} &= pU_n + qU_{n-1}, \\ V_{n+1} &= pV_n + qV_{n-1}, \end{aligned}$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$. It is clear that $V_n = pU_n + 2qU_{n-1}$.

When $p = q = 1$, $U_n = F_n$ (F_n denotes the nth Fibonacci number). When $p = 2$, $q = 1$, $U_n = \allowbreak = P_n$ (P_n denotes the nth Pell number). When $p = q = 1$, $V_n = L_n$ (L_n denotes the nth Lucas number).

Let α and β be the roots of the characteristic equation $x^2 - px - q = 0$. Then the sequences $\{U_n\}$ and $\{V_n\}$ have the following Binet's formulas

$$U_n = A_1\alpha^n + B_1\beta^n, \tag{1}$$

$$V_n = A_2\alpha^n + B_2\beta^n, \tag{2}$$

where

$$\alpha = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \beta = \frac{p - \sqrt{p^2 + 4q}}{2},$$

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$$A_1 = \frac{1}{\sqrt{p^2+4q}}, \quad B_1 = \frac{-1}{\sqrt{p^2+4q}},$$

$$A_2 = \frac{p-2\beta}{\sqrt{p^2+4q}}, \quad B_2 = \frac{2\alpha-p}{\sqrt{p^2+4q}}.$$

The third order linear recurrences of the Fibonacci and Lucas types are defined by the following equations:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

$$S_n = S_{n-1} + S_{n-2} + S_{n-3},$$

where $T_0 = 0$, $T_1 = T_2 = 1$ and $S_0 = 3$, $S_1 = 1$, $S_2 = 3$. Also, the sequences $\{T_n\}$ and $\{S_n\}$ are well known *Tribonacci sequences* with different initial conditions. It is clear that $S_n = T_n + 2T_{n-1} + 3T_{n-2}$.

Let γ_1, γ_2 and γ_3 be the roots of the characteristic equation $x^3 - x^2 - x - 1 = 0$. Then the sequences $\{T_n\}$ and $\{S_n\}$ have the following Binet's formulas

$$T_n = \frac{\gamma_1^{n+1}}{(\gamma_1-\gamma_2)(\gamma_1-\gamma_3)} + \frac{\gamma_2^{n+1}}{(\gamma_2-\gamma_1)(\gamma_2-\gamma_3)} + \frac{\gamma_3^{n+1}}{(\gamma_3-\gamma_1)(\gamma_3-\gamma_2)},$$

$$S_n = \gamma_1^n + \gamma_2^n + \gamma_3^n.$$

Some of the terms of the sequences $\{U_n\}$, $\{V_n\}$, $\{T_n\}$ and $\{S_n\}$ are the following:

n	0	1	2	3	4	5	6	7	8	9	10
U_n $_{p=2,q=3}$	0	1	2	7	20	61	182	547	1640	4921	14762
V_n $_{p=2,q=3}$	2	2	10	26	82	242	730	2186	6562	19682	59050
T_n	0	1	1	2	4	7	13	24	44	81	149
S_n	3	1	3	7	11	21	39	71	131	241	443

Some authors have studied second order or third order Fibonacci numbers and their certain generalizations $[11-13, 17-19]$. In [11], the authors have derived generalized Binet's formula and combinatorial representation of the generalized order k- Fibonacci numbers. For $p = A$ and $q = 1$, in [12] , the author has given the sums of squares of the terms of $\{U_n\}$ as follows:

$$\sum_{i=0}^{n} U_i^2 = \frac{U_n U_{n+1}}{A}.$$

In [13], the author has given the sums of the terms of Tribonacci sequence $\{T_n\}$ as follows:

$$\sum_{i=0}^{n} T_i = \frac{T_{n+2} + T_n - 1}{2}. \tag{3}$$

Similarly, by the induction method on n , we have

$$\sum_{i=0}^{n} S_i = \frac{S_{n+2} + S_n}{2}. \tag{4}$$

An $n \times n$ matrix C is called a *circulant matrix* if it is of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{bmatrix}.$$

For each $i, j = 1, 2, \ldots, n$ and $k = 0, 1, 2, \ldots, n-1$, all the elements (i, j) such that $j - i \equiv$ *k(mod n)* have the same value c_k; these elements form the so-called kth stripe of C. Obviously, a circulant matrix is determined by its first row (or column). That is $C = =$*Circ* $(c_0, c_1, \ldots, c_{n-1})$. The circulant matrices play important role in numerical analysis, because they can be quickly solved using the discrete Fourier transform.

Circulant matrices are especially tractable class of matrices since their inverses, conjugate transposes, sums and products also circulant. Moreover a circulant matrix is a normal matrix [5].

Many authors have studied circulant matrices [1, 3, 5, 6, 10, 26]. Hladnik [6] has given a formula for Schur norm of a block circulant matrix with circulant blocks. Karner et al. [10] have worked on spectral decompositions and singular value decompositions of four types of real circulant matrices. Bose and Mitra [3] have derived the limiting spectral distribution of a particular variant of a circulant random matrix. Atkin et al. [1] have studied the powers of a circulant. Zhang et al. [26] have worked on the minimal polynomials and inverses of a block circulant matrices over a field.

Matrix norms play important role in perturbation analysis, condition and error estimates [15, 16, 25]. Recently, there have been several papers on the norms of special circulant matrices [2, 4, 9, 14, 20 − 24]. Solak [21, 22] has defined $A = (a_{ij})$ and $B = (b_{ij})$ as $n \times n$ circulant matrices, where $a_{ij} \equiv F_{(\bmod\,(j-i,n))}$ and $b_{ij} \equiv L_{(\bmod\,(j-i,n))}$, then he has given some bounds for the spectral and Euclidean norms of the matrices A and B. Civciv and Türkmen [4] have constructed the circulant matrix with the Lucas number and presented lower and upper bounds for the Euclidean and spectral norms of this matrix. Bahsi and Solak [2] have defined $C_{a,r} = (c_{ij})$ as $n \times n$ circulant matrix where $c_{ij} \equiv a + (j - i \bmod n)\, r$, a and r are real numbers, then they have investigated eigenvalues, determinant, spectral norm, Euclidean norm of this matrix. Shen and Cen [20] have given upper and lower bounds for the spectral norms of r-circulant matrices in the forms $A = C_r\,(F_0, F_1, \ldots, F_{n-1})$, $B = C_r\,(L_0, L_1, \ldots, L_{n-1})$. Solak and Bozkurt [23] have defined almost circulant matrix as follows: $C_n = \mathrm{Circ}\left(a, 1, \frac{1}{2}, \ldots, \frac{1}{n-1}\right)$, where $a \in \mathbb{R}$ ($\mathbb{R}$ denotes the set of real numbers) and $a \neq 0$. After they have established upper bounds for the l_p norms of the matrix C_n. Ipek [9] have obtained the equality for the Solak's work in [21]. Kocer [14] has given some properties of the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers, then she has defined the circulant, negacyclic and semicirculant matrices with these numbers and she has investigated the norms, eigenvalues and determinants of these matrices.

In this paper, let $U =$*Circ* $(U_0, U_1, \ldots, U_{n-1})$, $V =$*Circ* $(V_0, V_1, \ldots, V_{n-1})$, $T =$ *Circ* $(T_0, T_1, \ldots, T_{n-1})$ and $S =$*Circ* $(S_0, S_1, \ldots, S_{n-1})$ be circulant matrices. Firstly we give equalities for the spectral norms of these matrices. After we obtain equalities and inequalities related to spectral norms of Hadamard and Kronecker product of these matrices.

Now we start with some preliminaries related to our study.

Definition 1.1. *Let $A = (a_{ij})$ be any $m \times n$ matrix. The spectral norm of the matrix A is*

$$\|A\|_2 = \sqrt{\max_i \lambda_i \left(A^H A\right)},$$

where $\lambda_i \left(A^H A\right)$ are eigenvalues of $A^H A$ and A^H is conjugate transpose of matrix A.

Definition 1.2. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ matrices. Then their Hadamard product $A \circ B$ is defined*

$$A \circ B = [a_{ij} b_{ij}] .$$

Definition 1.3. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be $m \times n$ and $p \times r$ matrices, respectively. Then their Kronecker product $A \otimes B$ is defined*

$$A \otimes B = [a_{ij} B] .$$

Lemma 1.1. *[8] Let A and B be $m \times n$ matrices. Then we have*

$$\|A \circ B\|_2 \leq \|A\|_2 \, \|B\|_2 .$$

Lemma 1.2. *[8] Let A and B be $m \times n$ matrices. Then we have*

$$\|A \otimes B\|_2 = \|A\|_2 \, \|B\|_2 .$$

Lemma 1.3. *[7] Let A be any $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of $A^H A$ are $|\lambda_1|^2, |\lambda_2|^2, \ldots, |\lambda_n|^2$.*

Lemma 1.4. *[10] Let $C =$Circ $(c_0, c_1, \ldots, c_{n-1})$ be $n \times n$ general circulant matrix. Then*

$$\underset{0 \leq m \leq n-1}{\lambda_m} = \sum_{k=0}^{n-1} c_k w^{-mk}, \tag{5}$$

where λ_j are eigenvalues of C and w is the nth primitive root of unity.

Let w be $w = e^{\frac{2\pi i}{n}}$. Then w is a nth primitive root of unity. Then the equality (5) has the form

$$\underset{0 \leq m \leq n-1}{\lambda_m} = \sum_{k=0}^{n-1} c_k e^{\frac{-2\pi i m k}{n}} . \tag{6}$$

2. Main results

Theorem 2.1. *The spectral norm of matrix $U =$Circ $(U_0, U_1, \ldots, U_{n-1})$ is*

$$\|U\|_2 = \frac{1 - U_n - qU_{n-1}}{1 - p - q}.$$

Proof. Since U is a circulant matrix, from (6) its eigenvalues are of the form

$$\underset{0 \leq m \leq n-1}{\lambda_m} = \sum_{k=0}^{n-1} U_k e^{\frac{-2\pi i m k}{n}} .$$

Then for $m = 0$, using the Binet's formula for the sequence $\{U_n\}$, we have

$$\begin{aligned}
\lambda_0 &= \sum_{k=0}^{n-1} U_k = \sum_{k=0}^{n-1} \left(A_1 \alpha^k + B_1 \beta^k\right) = \sum_{k=0}^{n-1} A_1 \alpha^k + \sum_{k=0}^{n-1} B_1 \beta^k = \\
&= \frac{A_1 \alpha^n - A_1}{\alpha - 1} + \frac{B_1 \beta^n - B_1}{\beta - 1} = \\
&= \frac{\alpha\beta \left(A_1 \alpha^{n-1} + B_1 \beta^{n-1}\right) - (A_1 \alpha^n + B_1 \beta^n) - (A_1 \beta + B_1 \alpha) + A_1 + B_1}{\alpha\beta - (\alpha + \beta) + 1}.
\end{aligned}$$

Since, $\alpha+\beta=p$, $\alpha\beta=-q$, $A_1+B_1=0$ and $A_1\beta+B_1\alpha=-1$, we have

$$\lambda_0=\sum_{k=0}^{n-1}U_k=\frac{1-U_n-qU_{n-1}}{1-p-q}. \tag{7}$$

On the other hand, we have

$$\underset{1\le m\le n-1}{|\lambda_m|}=\left|\sum_{k=0}^{n-1}U_k e^{\frac{-2\pi imk}{n}}\right|\le\sum_{k=0}^{n-1}|U_k|\left|e^{\frac{-2\pi imk}{n}}\right|\le\sum_{k=0}^{n-1}|U_k|=\sum_{k=0}^{n-1}U_k. \tag{8}$$

Using the Lemma 1.3 and the fact that a circulant matrix is a normal, we have

$$\|U\|_2=\max_{0\le m\le n-1}|\lambda_m|=\max\left(|\lambda_0|,\max_{1\le m\le n-1}|\lambda_m|\right). \tag{9}$$

Finally, from (7), (8) and (9), we have

$$\|U\|_2=\frac{1-U_n-qU_{n-1}}{1-p-q}.$$

Thus the proof is completed. □

Theorem 2.2. *The spectral norm of matrix $V=$Circ $(V_0,V_1,\ldots,V_{n-1})$ is*

$$\|V\|_2=\frac{2-p-V_n-qV_{n-1}}{1-p-q}.$$

Proof. Using the Binet's formula for the sequence $\{V_n\}$, we have

$$\begin{aligned}\sum_{k=0}^{n-1}V_k &= \sum_{k=0}^{n-1}\left(A_2\alpha^k+B_2\beta^k\right)=\sum_{k=0}^{n-1}A_2\alpha^k+\sum_{k=0}^{n-1}B_2\beta^k=\\ &= \frac{A_2\alpha^n-A_2}{\alpha-1}+\frac{B_2\beta^n-B_2}{\beta-1}=\\ &= \frac{\alpha\beta\left(A_2\alpha^{n-1}+B_2\beta^{n-1}\right)-\left(A_2\alpha^n+B_2\beta^n\right)-\left(A_2\beta+B_2\alpha\right)+A_2+B_2}{\alpha\beta-(\alpha+\beta)+1}.\end{aligned}$$

Since, $\alpha+\beta=p$, $\alpha\beta=-q$, $A_2+B_2=2$ and $A_2\beta+B_2\alpha=p$, we have

$$\sum_{k=0}^{n-1}V_k=\frac{2-p-V_n-qV_{n-1}}{1-p-q}. \tag{10}$$

Since V is a circulant matrix, from (6) its eigenvalues are of the form

$$\underset{0\le m\le n-1}{\lambda_m}=\sum_{k=0}^{n-1}V_k e^{\frac{-2\pi imk}{n}}.$$

Then for $m=0$, using (10) we have

$$\lambda_0=\sum_{k=0}^{n-1}V_k=\frac{2-p-V_n-qV_{n-1}}{1-p-q}. \tag{11}$$

From Lemma 1.3 and the fact that the matrix V is a normal matrix, we have

$$\|V\|_2=\max_{0\le m\le n-1}|\lambda_m|=\max\left(|\lambda_0|,\max_{1\le m\le n-1}|\lambda_m|\right). \tag{12}$$

Since

$$\underset{1\le m\le n-1}{|\lambda_m|}=\left|\sum_{k=0}^{n-1}V_k e^{\frac{-2\pi imk}{n}}\right|\le\sum_{k=0}^{n-1}|V_k|\left|e^{\frac{-2\pi imk}{n}}\right|\le\sum_{k=0}^{n-1}|V_k|=\sum_{k=0}^{n-1}V_k, \tag{13}$$

from (11), (12) and (13), we have

$$\|V\|_2 = \frac{2-p-V_n-qV_{n-1}}{1-p-q}.$$

Then the proof is completed. □

When $p=q=1$, $U_n=F_n$ (F_n denotes the nth Fibonacci number) and $V_n=L_n$ (L_n denotes the nth Lucas number). Then by Theorems 2.1 and 2.2, we have

$$\|U\|_2 = F_{n+1}-1 \quad \text{and} \quad \|V\|_2 = L_{n+1}-1.$$

In fact, these equalities are the spectral norms of circulant matrix with the Fibonacci and Lucas numbers.

Corollary 2.1. *For $n \geq 2$, the spectral norms of $V_{n\times n} = V = Circ\,(V_0, V_1, \ldots, V_{n-1})$ and $U_{n\times n} = U = Circ\,(U_0, U_1, \ldots, U_{n-1})$ have the following equality*

$$\|V_{n\times n}\|_2 = p\,\|U_{n\times n}\|_2 + 2q\left\|U_{(n-1)\times(n-1)}\right\|_2 + 2,$$

where $U_{(n-1)\times(n-1)} = Circ\,(U_0, U_1, \ldots, U_{n-2})$.

Proof. Since $V_n = pU_n + 2qU_{n-1}$, the proof is trivial from Theorems 2.1 and 2.2. □

Corollary 2.2. *The spectral norm of the Hadamard product of $U = Circ\,(U_0, U_1, \ldots, U_{n-1})$ and $V = Circ\,(V_0, V_1, \ldots, V_{n-1})$ has the following inequality*

$$\|U \circ V\|_2 \leq \frac{(1-U_n-qU_{n-1})(2-p-V_n-qV_{n-1})}{(1-p-q)^2}.$$

Proof. Since $\|U\circ V\|_2 \leq \|U\|_2\,\|V\|_2$, the proof is trivial from Theorems 2.1 and 2.2. □

Corollary 2.3. *The spectral norm of the Kronecker product of $U = Circ\,(U_0, U_1, \ldots, U_{n-1})$ and $V = Circ\,(V_0, V_1, \ldots, V_{n-1})$ has the following equality*

$$\|U \otimes V\|_2 = \frac{(1-U_n-qU_{n-1})(2-p-V_n-qV_{n-1})}{(1-p-q)^2}.$$

Proof. Since $\|U\otimes V\|_2 = \|U\|_2\,\|V\|_2$, the proof is trivial from Theorems 2.1 and 2.2. □

Theorem 2.3. *The spectral norm of matrix $T = Circ\,(T_0, T_1, \ldots, T_{n-1})$ is*

$$\|T\|_2 = \frac{T_{n+1}+T_{n-1}-1}{2}.$$

Proof. Since T is a circulant matrix, from (6) its eigenvalues are of the form

$$\lambda_{\substack{m\\0\leq m\leq n-1}} = \sum_{k=0}^{n-1} T_k e^{\frac{-2\pi i m k}{n}}.$$

Then for $m=0$, using (3) we have

$$\lambda_0 = \sum_{k=0}^{n-1} T_k = \frac{T_{n+1}+T_{n-1}-1}{2}. \tag{14}$$

On the other hand, we have

$$\left|\lambda_m\right|_{1\leq m\leq n-1} = \left|\sum_{k=0}^{n-1} T_k e^{\frac{-2\pi i m k}{n}}\right| \leq \sum_{k=0}^{n-1} |T_k| \left|e^{\frac{-2\pi i m k}{n}}\right| \leq \sum_{k=0}^{n-1} |T_k| = \sum_{k=0}^{n-1} T_k. \tag{15}$$

Using Lemma 1.3 and the fact that the matrix T is a normal matrix, we have

$$\|T\|_2 = \max_{0\le m\le n-1} |\lambda_m| = \max\left(|\lambda_0|, \max_{1\le m\le n-1} |\lambda_m|\right). \tag{16}$$

From (14), (15) and (16), we have

$$\|T\|_2 = \frac{T_{n+1} + T_{n-1} - 1}{2}.$$

Therefore the proof is completed. □

Theorem 2.4. *The spectral norm of matrix $S = Circ\ (S_0, S_1, \ldots, S_{n-1})$ is*

$$\|S\|_2 = \frac{S_{n+1} + S_{n-1}}{2}.$$

Proof. Since S is a circulant matrix, from (6) its eigenvalues are of the form

$$\lambda_{\substack{m\\0\le m\le n-1}} = \sum_{k=0}^{n-1} S_k e^{\frac{-2\pi imk}{n}}.$$

Then for $m = 0$, using (4) we have

$$\lambda_0 = \sum_{k=0}^{n-1} S_k = \frac{S_{n+1} + S_{n-1}}{2}. \tag{17}$$

From Lemma 1.3 and the fact that the matrix S is a normal matrix, we have

$$\|S\|_2 = \max_{0\le m\le n-1} |\lambda_m| = \max\left(|\lambda_0|, \max_{1\le m\le n-1} |\lambda_m|\right). \tag{18}$$

Since

$$\underset{1\le m\le n-1}{|\lambda_m|} = \left|\sum_{k=0}^{n-1} S_k e^{\frac{-2\pi imk}{n}}\right| \le \sum_{k=0}^{n-1} |S_k| \left|e^{\frac{-2\pi imk}{n}}\right| \le \sum_{k=0}^{n-1} |S_k| = \sum_{k=0}^{n-1} S_k, \tag{19}$$

from (17), (18) and (19), we have

$$\|S\|_2 = \frac{S_{n+1} + S_{n-1}}{2}.$$

Thus the proof is completed. □

Corollary 2.4. *For $n \ge 3$, the spectral norms of $T_{n\times n} = T = Circ\ (T_0, T_1, \ldots, T_{n-1})$ and $S_{n\times n} = S = Circ\ (S_0, S_1, \ldots, S_{n-1})$ have the following equality*

$$\|S_{n\times n}\|_2 = \|T_{n\times n}\|_2 + 2\left\|T_{(n-1)\times(n-1)}\right\|_2 + 3\left\|T_{(n-2)\times(n-2)}\right\|_2 + 3,$$

where $T_{(n-1)\times(n-1)} = Circ\ (T_0, T_1, \ldots, T_{n-2})$ and $T_{(n-2)\times(n-2)} = Circ\ (T_0, T_1, \ldots, T_{n-3})$.

Proof. Since $S_n = T_n + 2T_{n-1} + 3T_{n-2}$, the proof is trivial from Theorems 2.3 and 2.4. □

Corollary 2.5. *The spectral norm of the Hadamard product of $T = Circ\ (T_0, T_1, \ldots, T_{n-1})$ and $S = Circ\ (S_0, S_1, \ldots, S_{n-1})$ has the following inequality*

$$\|T \circ S\|_2 \le \frac{(T_{n+1} + T_{n-1} - 1)(S_{n+1} + S_{n-1})}{4}.$$

Proof. Since $\|T \circ S\|_2 \le \|T\|_2 \|S\|_2$, the proof is trivial from Theorems 2.3 and 2.4. □

Corollary 2.6. *The spectral norm of the Kronecker product of $T = Circ\ (T_0, T_1, \ldots, T_{n-1})$ and $S = Circ\ (S_0, S_1, \ldots, S_{n-1})$ has the following equality*

$$\|T \otimes S\|_2 = \frac{(T_{n+1} + T_{n-1} - 1)(S_{n+1} + S_{n-1})}{4}.$$

Proof. Since $\|T \otimes S\|_2 = \|T\|_2 \|S\|_2$, the proof is trivial from Theorems 2.3 and 2.4. □

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